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$N = 2$ topological Yang–Mills theories and Donaldson's polynomials

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Abstract

The $N = 2$ topological Yang–Mills and holomorphic Yang–Mills theories on simply connected compact Kähler surfaces with $p_g \geq 1$ are re-examined. The $N = 2$ symmetry is clarified in terms of a Dolbeault model of the equivariant cohomology. We realize the non-algebraic part of Donaldson's polynomial invariants as well as the algebraic part. We calculate Donaldson's polynomials on $H^{2,0}(S, \mathbb{Z}) \oplus H^{0,2}(S, \mathbb{Z})$.

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1. Introduction

The $N = 2$ super Yang–Mills theory on arbitrary four-manifolds can be twisted to define $N = 1$ topological Yang–Mills (TYM) theory which realize Donaldson's polynomial invariants of smooth four-manifolds [1,2] as correlation functions [3]. Recently, Witten has determined the Donaldson invariants of compact Kähler surfaces with $p_g \geq 1$ by exploiting some standard properties of $N = 2$ and $N = 1$ super Yang–Mills theories [4].

Some time ago, Park [5] proposed $N = 2$ TYM theory on compact Kähler surfaces. His construction is based directly on the $N = 1$ TYM theory utilizing the complex and Kähler structures of the moduli space of anti-self dual (ASD) connections. He has also proposed

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$N = 2$ holomorphic Yang–Mills (HYM) theory whose partition function is a generating functional of certain Donaldson invariants [6], adapting the two-dimensional construction of Witten's to Kähler surfaces [7]. However, both theories describe the algebraic part of the Donaldson invariants (the Donaldson invariants depending on $H^{1,1}(X, \mathbb{Z})$), analogous to the invariants defined by Li [8], rather than all the invariants and the non-algebraic part was simply ignored. Furthermore, we will see that it is impossible to realize the non-algebraic part in those constructions. The purpose of this paper is to fill those gaps.

In this paper, we re-examine $N = 2$ TYM and HYM theories on simply connected compact Kähler surfaces with $p_g \geq 1$, which lead to the different $N = 2$ (global) supersymmetry transformation laws for some auxiliary fields. This allows us to realize the non-algebraic part of Donaldson's polynomials as well as the algebraic part. We calculate Donaldson's polynomial invariants on $H^{2,0}(X, \mathbb{Z}) \oplus H^{0,2}(X, \mathbb{Z})$.

This paper is organized as follows: in Section 2, we give backgrounds and motivations of this paper. We compare the basic supersymmetry transformation laws of the $N = 1$ and the $N = 2$ TYM theories in terms of the de Rham and a Dolbeault models of the equivariant cohomology. We show that the Dolbeault equivariant cohomology is not isomorphic to the de Rham equivariant cohomology. In the field theoretical context, this amounts to introducing on-shell observables in the $N = 2$ TYM and HYM theories. In Section 3, we construct new $N = 2$ TYM theory. We briefly discuss the geometrical and the physical meanings of fermionic zero-modes. We resolve the problem of the on-shell invariants adapting Witten's method of introducing the mass gap [4]. In Section 4, we study deformations to HYM theories and calculate Donaldson's invariants on $H^{2,0}(X, \mathbb{Z}) \oplus H^{0,2}(X, \mathbb{Z})$. We also show that the broken part of the $N = 2$ supersymmetry due to the mass gap is restored in the process of the deformation. We compare our results with others and give some general remarks on the algebraic parts of the invariants. Our method will lead us to determine the full invariants for simply connected $K3$ surfaces.

2. Backgrounds and motivations

We consider a simply connected compact Kähler surface X with Kähler form ω and $b_2^+ = 1 + 2p_g \geq 3$ where b_2^+ and p_g denote the number of the self-dual harmonic two forms and the geometric genus, respectively. Let E be a complex vector bundle over X with the restriction of structure group to $SU(2)$. We write $\mathfrak{g}_E$ for the Lie algebra bundle associated with E by adjoint representation. We introduce a positive definite quadratic form $(a, b) = -\mathrm{Tr}\, ab$ on $\mathfrak{su}(2)$, where Tr denotes the trace in the two-dimensional representation. Then, the bundle E is classified by the instanton number

$$k = \langle c_2(E), X\rangle = \frac{1}{8\pi^2}\int_X \mathrm{Tr}\, F \wedge F \in \mathbb{Z}.$$

Let $\mathcal{A}$ denote the space of all connections, which is an affine space whose tangent vectors are represented by $\mathfrak{g}_E$-valued one form $\delta A \in \Omega^1(\mathfrak{g}_E)$. Let $\mathcal{G}$ be the group of gauge transformations.

2.1. The $N = 1$ supersymmetry

The global supersymmetry operator δ_w of the $N = 1$ topological Yang–Mills theory can be interpreted as the exterior (covariant) derivative on $\mathcal{A}/\mathcal{G}$ [3,9]. The $N = 1$ supersymmetry transformation laws for the basic multiplet (A, Ψ, Φ) are

$$\delta_w A = -\Psi, \qquad \delta_w \Psi = -\mathrm{i} d_A \Phi, \qquad \delta_w \Phi = 0, \tag{2.1}$$

where $\Psi \in \Omega^1(\mathfrak{g}_E)$ and $\Phi \in \Omega^0(\mathfrak{g}_E)$. One introduces a global quantum number (or the ghost number) U which assigns the value 1 to δ_w. The U numbers of the basic fields (A, Ψ, Φ) are (0, 1, 2). Note that $\delta_w^2 = -\mathrm{i}\delta_\Phi$, where δ_Φ is the generator of a gauge transformation with infinitesimal parameter Φ. Thus, $\delta_w^2 = 0$ if it acts on a $\mathcal{G}$-invariant functional of the basic fields. The supersymmetry operator δ_w can be viewed as the de Rham cohomology operator on $\mathcal{A}/\mathcal{G}$ if $\mathcal{G}$ acts freely on $\mathcal{A}$.

More precisely, δ_w is the operator of the de Rham model for the $\mathcal{G}$-equivariant cohomology of $\mathcal{A}$.[2] Let Lie($\mathcal{G}$) be the Lie algebra of $\mathcal{G}$ which is the space $\Omega^0(\mathfrak{g}_E)$ of $\mathfrak{g}_E$-valued zero-forms. The $\mathcal{G}$-action on $\mathcal{A}$ is generated by vector fields V_a, where we pick an orthonormal basis T_a of Lie($\mathcal{G}$). Let Fun(Lie($\mathcal{G}$)) be the algebra of polynomial functions, generated by Φ^a with degree 2, on Lie($\mathcal{G}$). The $\mathcal{G}$-equivariant de Rham complex is $\Omega^*_{\mathcal{G}}(\mathcal{A}) = (\Omega^*(\mathcal{A}) \otimes \mathrm{Fun}(\mathrm{Lie}(\mathcal{G})))^{\mathcal{G}}$. The associated differential operator is δ_w which can be represented as

$$\delta_w = -\sum_I \Psi^I \frac{\partial}{\partial A^I} + \mathrm{i}\sum_{I,a} \varphi^a V_a^I \frac{\partial}{\partial \Psi^I}, \tag{2.2}$$

where A^I are the local coordinates on $\mathcal{A}$. We have

$$\delta_w^2 = -\mathrm{i}\Phi^a \mathcal{L}_a, \tag{2.3}$$

where $\mathcal{L}_a$ is the Lie derivative with respect to V_a. Thus, $\delta_w^2 = 0$ on the $\mathcal{G}$-invariant subspace $\Omega^*_{\mathcal{G}}(\mathcal{A})$ of $\Omega^*(\mathcal{A}) \otimes \mathrm{Fun}(\mathrm{Lie}(\mathcal{G}))$. The $\mathcal{G}$-equivariant de Rham cohomology $H^*_{\mathcal{G}}(\mathcal{A})$ is defined as the pairs $(\Omega^*_{\mathcal{G}}(\mathcal{A}),\ \delta_w)$.

In Donaldson–Witten theory, we are interested in the $\mathcal{G}$-equivariant cohomology of the space of anti-self dual (ASD) connections. Since there are no reducible ASD connections, for generic metrics on X, $\mathcal{G}$ acts freely on the space of ASD connections, the $\mathcal{G}$-equivariant cohomology reduces to the de Rham cohomology of the moduli space $\mathcal{M}$ of ASD connections. The de Rham cohomology on $\mathcal{M}$ can be obtained from $H^*_{\mathcal{G}}(\mathcal{A})$ by restriction and reduction. For example, an element of $H^2_{\mathcal{G}}(\mathcal{A})$ is given by

$$\tilde{\omega}^{(2)} = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(\Psi \wedge \Psi + \mathrm{i}\Phi F_A) \wedge \omega^{(2)}, \tag{2.4}$$

where $\omega^{(2)} \in H^2(X, \mathbb{Z})$. The cohomology class of $H^2_{\mathcal{G}}(\mathcal{A})$ depends only on the cohomology class of $H^2(X, \mathbb{Z})$. In Witten's approach, an element of $H^2(\mathcal{M})$ can be obtained from $H^2_{\mathcal{G}}(\mathcal{A})$ by the field theoretical methods, in which Ψ is eventually replaced by its zero-modes, A by

[2] We refer the reader to [7,10–12] for details on the equivariant cohomology. We generally follow [7].

the ASD connection and Φ by its vacuum expectation value. One can also view $\tilde{\omega}^{(2)}$ as the equivariantly closed extension [10] of a closed form $(1/8\pi^2)\int_X \mathrm{Tr}(\Psi \wedge \Psi) \wedge \omega^{(2)}$ on $\mathcal{A}$. It is also known that any element of $H^*(\mathcal{M})$ is induced from an element of $H^*_{\mathcal{G}}(\mathcal{A})$ [2,13].

2.2. The $N = 2$ supersymmetry

Picking a complex structure J on X, one can introduce a complex structure $J_{\mathcal{A}}$ on $\mathcal{A}$ as well as on $\mathcal{A}/\mathcal{G}$,

$$J_{\mathcal{A}}\delta A = J\delta A\,, \quad \delta A \in T\mathcal{A} \tag{2.5}$$

by identifying $T^{1,0}\mathcal{A}$ and $T^{0,1}\mathcal{A}$ in $T\mathcal{A} = T^{1,0}\mathcal{A} \oplus T^{0,1}\mathcal{A}$ with the $\mathfrak{g}_E$ valued $(1,0)$-forms and $(0,1)$-forms on X, respectively. We can also introduce natural Kähler structure on $\mathcal{A}$ with Kähler form

$$\tilde{\omega} = \frac{1}{8\pi^2}\int\limits_X \mathrm{Tr}(\delta A \wedge \delta A) \wedge \omega. \tag{2.6}$$

Using the complex structures J and $J_{\mathcal{A}}$, we can decompose $\delta_W = \mathbf{s} + \bar{\mathbf{s}}$ and find the $N = 2$ transformation laws for the basic multiplet $(A', A'', \psi, \bar{\psi}, \varphi)$ [5]:

$$\begin{array}{lll} \mathbf{s}A' = -\psi, & \mathbf{s}\psi = 0, & \\ \bar{\mathbf{s}}A' = 0, & \bar{\mathbf{s}}\psi = -\mathrm{i}\partial_A\varphi, & \bar{\mathbf{s}}\varphi = 0, \\ \mathbf{s}A'' = 0, & \mathbf{s}\bar{\psi} = -\mathrm{i}\bar{\partial}_A\varphi, & \mathbf{s}\varphi = 0, \\ \bar{\mathbf{s}}A'' = -\bar{\psi}, & \bar{\mathbf{s}}\bar{\psi} = 0, & \end{array} \tag{2.7}$$

where $\psi \in \Omega^{1,0}(\mathfrak{g}_E)$, $\bar{\psi} \in \Omega^{0,1}(\mathfrak{g}_E)$ and $\varphi \in \Omega^{0,0}(\mathfrak{g}_E)$. Note that ψ can be identified with holomorphic (co)tangent vectors on $\mathcal{A}$. It is important to note that φ is of degree $(1,1)$. We introduce two global quantum numbers (or ghost numbers) (U, R), which assign $(1,1)$ to $\mathbf{s}$ and $(1,-1)$ to $\bar{\mathbf{s}}$. A quantity of degree (p,q) has $U = p+q$ and $R = p-q$. The above transformation laws play a central role in constructing $N = 2$ TYM theories.

The commutation relations of the fermionic symmetry generators $\mathbf{s}, \bar{\mathbf{s}}$ are

$$\mathbf{s}^2 = 0, \qquad (\mathbf{s}\bar{\mathbf{s}} + \bar{\mathbf{s}}\mathbf{s}) = \mathrm{i}d_A\varphi = -\mathrm{i}\delta_\varphi, \qquad \bar{\mathbf{s}}^2 = 0, \tag{2.8}$$

where δ_φ is the generator of a gauge transformation with infinitesimal parameter φ. Thus $\{\mathbf{s}, \bar{\mathbf{s}}\} = 0$ precisely on the $\mathcal{G}$-invariant space or if it acts on $\mathcal{G}$-invariant functionals of $A', A'', \psi, \bar{\psi}, \varphi$. Thus, $\bar{\mathbf{s}}$ can be roughly viewed as the operator of Dolbeault cohomology group on $\mathcal{A}/\mathcal{G}$.

In fact, $\bar{\mathbf{s}}$ is the operator of a Dolbeault cohomological analogue of $\mathcal{G}$-equivariant cohomology of $\mathcal{A}$. [3] This can be formally described as follows: we let $\Omega^{*,*}(\mathcal{A})$ be the Dolbeault complex on $\mathcal{A}$. Now we interpret Fun(Lie($\mathcal{G}$)) to the algebra of polynomials functions generated by φ^a. Then the desired Dolbeault model of the $\mathcal{G}$-equivariant complex is $\Omega^{*,*}_{\mathcal{G}} =$

[3] Note that the transformations (2.7) are slightly different, *in convention*, from those in [5]. Here we follow the usual conventions of physics literature.

$(\Omega^{*,*}(\mathcal{A}) \otimes \mathrm{Fun}(\mathcal{G}))^{\mathcal{G}}$. The associated differential operators with the degrees (1, 0) and (0, 1) are s and $\bar{\mathrm{s}}$ represented by

$$
\begin{aligned}
\mathrm{s} &= -\sum_i \psi^i \frac{\partial}{\partial A'^i} + \mathrm{i}\sum_{\bar{i},a} \varphi^a V_a^{\bar{i}} \frac{\partial}{\partial \bar{\psi}^{\bar{i}}}, \\
\bar{\mathrm{s}} &= -\sum_{\bar{i}} \bar{\psi}^{\bar{i}} \frac{\partial}{\partial A''^{\bar{i}}} + \mathrm{i}\sum_{i,a} \varphi^a V_a^{i} \frac{\partial}{\partial \psi^{i}},
\end{aligned}
\tag{2.9}
$$

where $i, \bar{i}$ are the local holomorphic and anti-holomorphic indices tangent to $\mathcal{A}$. We have

$$
\mathrm{s}^2 = 0, \qquad \mathrm{s}\bar{\mathrm{s}} + \bar{\mathrm{s}}\mathrm{s} = -\mathrm{i}\varphi^a \mathcal{L}_a, \qquad \bar{\mathrm{s}}^2 = 0. \tag{2.10}
$$

Thus, $\{\mathrm{s}, \bar{\mathrm{s}}\} = 0$ on the $\mathcal{G}$-invariant subspace $\Omega_{\mathcal{G}}^{*,*}$ of $\Omega^{*,*}(\mathcal{A}) \otimes \mathrm{Fun}(\mathrm{Lie}(\mathcal{G}))$.

We call elements of $\Omega_{\mathcal{G}}^{*,*}(\mathcal{A})$ such that $\bar{\mathrm{s}}\alpha = 0$ (equivariantly) $\bar{\mathrm{s}}$-closed forms and those of the form $\alpha = \bar{\mathrm{s}}\beta$ (equivariantly) $\bar{\mathrm{s}}$-exact forms. Since $\bar{\mathrm{s}}$ defines a map $\bar{\mathrm{s}} : \Omega_{\mathcal{G}}^{*,*}(\mathcal{A}) \to \Omega_{\mathcal{G}}^{*,*+1}(\mathcal{A})$ and $\bar{\mathrm{s}}^2 = 0$ for any $\alpha \in \Omega_{\mathcal{G}}^{*,*}(\mathcal{A})$, the pairs $(\Omega_{\mathcal{G}}^{*,*}(\mathcal{A}), \bar{\mathrm{s}})$ is a complex. *We define the $\mathcal{G}$-equivariant Dolbeault cohomology $H_{\mathcal{G}}^{*,*}(\mathcal{A})$ by the cohomology of the complex $(\Omega_{\mathcal{G}}^{*,*}(\mathcal{A}), \bar{\mathrm{s}})$.*

An immediate observation is that the analogue of the Hodge decomposition theorem will not be applicable in the equivariant sense. Since $\mathcal{A}$ has the Kähler structure, the de Rham and the Dolbeault cohomologies on $\mathcal{A}$ are related by the Hodge decompositions. If we assume $\mathcal{G}$ acts freely on $\mathcal{A}$, we can expect our equivariant Dolbeault cohomology $\Omega_{\mathcal{G}}^{*,*}(\mathcal{A})$ is isomorphic to the usual Dolbeault cohomology on $\mathcal{A}/\mathcal{G}$; and the equivariant de Rham cohomology $\Omega_{\mathcal{G}}^{*}(\mathcal{A})$ is isomorphic to the de Rham cohomology of $\mathcal{A}/\mathcal{G}$. Since the Kähler structure on $\mathcal{A}$ does not descend to $\mathcal{A}/\mathcal{G}$ in general, the Hodge decomposition theorem is not applicable in general. That is, a $\mathcal{G}$-invariant and $\bar{\mathrm{s}}$-closed quantity is not automatically s-closed one.

2.3. *The old construction*

In the old construction [5], we introduced an anti-ghost B, a self-dual two form $B = B^{2,0} + B^{0,2} + B^0\omega \in \Omega_+^2(\mathfrak{g}_E)$ in the adjoint representation, with $(U, R) = (-2, 0)$. Then (2.8) naturally leads us to the multiplet $(B, \mathrm{i}\chi, -\mathrm{i}\bar{\chi}, H)$ with transformation laws

$$
\begin{aligned}
\mathrm{s}B &= -\mathrm{i}\chi, & \mathrm{s}\chi &= 0, \\
\bar{\mathrm{s}}B &= \mathrm{i}\bar{\chi}, & \bar{\mathrm{s}}\bar{\chi} &= 0, \\
\mathrm{s}\bar{\chi} &= H - \tfrac{1}{2}[\varphi, B], & \mathrm{s}H &= -\tfrac{1}{2}\mathrm{i}[\varphi, \chi], \\
\bar{\mathrm{s}}\chi &= H + \tfrac{1}{2}[\varphi, B], & \bar{\mathrm{s}}H &= -\tfrac{1}{2}\mathrm{i}[\varphi, \bar{\chi}].
\end{aligned}
\tag{2.11}
$$

The ghost numbers of the various fields are given by

(2.12)

Fields	A'	A''	ψ	$\bar{\psi}$	φ	B	χ	$\bar{\chi}$	H
U Number	0	0	1	1	2	−2	−1	−1	0
R Number	0	0	1	−1	0	0	1	−1	0

The action of $N = 2$ TYM theory can be written in the form [4]

$$S_{\text{old}} = \frac{\mathrm{s}\bar{\mathrm{s}} - \mathrm{s}\bar{\mathrm{s}}}{2}\mathcal{B}_{\mathrm{T}} = \frac{\mathrm{s}\bar{\mathrm{s}} - \bar{\mathrm{s}}\mathrm{s}}{2}\left(-\frac{1}{h^2}\int_X \mathrm{Tr}\, B \wedge *F - \frac{1}{h^2}\int_X \mathrm{Tr}\,\chi \wedge *\bar{\chi}\right). \tag{2.13}$$

Note that V has $(U, R) = (-2, 0)$, so that the action has $(U, R) = (0, 0)$. We find that

$$\begin{aligned} S_{\text{old}} = \frac{1}{h^2}\int_X \mathrm{Tr}\Big[&-H^{2,0} \wedge *(H^{0,2} + \mathrm{i}F^{0,2}) - H^{0,2} \wedge *(H^{2,0} + \mathrm{i}F^{2,0}) \\ &+\mathrm{i}\chi^{2,0} \wedge *\bar{\partial}_A\bar{\psi} + \mathrm{i}\bar{\chi}^{0,2} \wedge *\partial_A\psi + \mathrm{i}[\varphi, \chi^{2,0}] \wedge *\bar{\chi}^{0,2} \\ &+\mathrm{i}[\varphi, \chi^{0,2}] \wedge *\bar{\chi}^{2,0} - \tfrac{1}{2}\mathrm{i}B^{2,0} \wedge *\bar{\partial}_A\bar{\partial}_A\varphi + \tfrac{1}{2}\mathrm{i}B^{0,2} \wedge *\partial_A\partial_A\varphi \\ &+\tfrac{1}{2}[\varphi, B^{2,0}] \wedge *[\varphi, B^{0,2}] - \Big(2H^0(H^0 + \mathrm{i}f) - \mathrm{i}\bar{\chi}^0 \Lambda\bar{\partial}_A\psi \\ &-\mathrm{i}\chi^0\Lambda\partial_A\bar{\psi} - 2\mathrm{i}[\varphi, \chi^0]\bar{\chi}^0 - \tfrac{1}{2}[\varphi, B^0][\varphi, B^0] \\ &+\tfrac{1}{2}B^0\Lambda\left((\mathrm{i}\partial_A\bar{\partial}_A - \mathrm{i}\bar{\partial}_A\partial_A)\varphi - 2[\psi, \bar{\psi}]\right)\Big)\frac{\omega^2}{2!}\Big], \end{aligned} \tag{2.14}$$

where $f = \frac{1}{2}\Lambda F$ and Λ is adjoint to the wedge multiplication of ω.

For the details how $N = 2$ TYM theory (or TYM theory in general) realizes the Donaldson invariants, we refer the reader to [5] ([3,14]). We will show in Section 2.4 that the old $N = 2$ TYM theory realizes the algebraic part of Donaldson's polynomials only.

2.4. *Problem of the on-shell invariants*

An observable of $N = 2$ supersymmetric TYM theory should be gauge invariant as well as invariant under s and $\bar{\mathrm{s}}$. The candidates of the non-trivial topological observables depending on $H^2(X, \mathbb{Z})$ are

$$\begin{aligned} \tilde{\omega}^{2,0} &= \frac{1}{8\pi^2}\int_X \mathrm{Tr}(\psi \wedge \psi) \wedge \omega^{0,2}, \\ \tilde{\omega}^{0,2} &= \frac{1}{8\pi^2}\int_X \mathrm{Tr}(\bar{\psi} \wedge \bar{\psi}) \wedge \omega^{2,0}, \\ \tilde{\omega}^{1,1} &= \frac{1}{4\pi^2}\int_X \mathrm{Tr}(\mathrm{i}\varphi F^{1,1} + \psi \wedge \bar{\psi}) \wedge \omega^{1,1}, \end{aligned} \tag{2.15}$$

[4] The action of $N = 1$ TYM theory can be written as $S = -\mathrm{i}\delta_{\mathcal{W}} W$ [3,14]. The relation between $N = 1$ and $N = 2$ theories can be most conveniently understood with an analogy to the exterior derivative $d = \partial + \bar{\partial}$. An exact real (p, p)-form $\alpha = d\beta$ on a compact Kähler manifold can be written as

$$\alpha = \tfrac{1}{2}(dd^c)\gamma = \tfrac{1}{2}(\mathrm{i}\partial\bar{\partial} - \mathrm{i}\bar{\partial}\partial)\gamma = \mathrm{i}\partial\bar{\partial}\gamma$$

for some $(p-1, p-1)$-form ρ, where $d^c \equiv -J^{-1}dJ = \mathrm{i}(\bar{\partial} - \partial)$.

where $\omega^{p,q} \in H^{p,q}(X, \mathbb{Z})$ and we generally denote $\tilde{\omega}^{r,s}$ as an (r, s)-form on $\mathcal{A}$ (or of degree (r, s)). Note that the above quantities are the components of the decompositions of $\tilde{\omega}^{(2)} \in H^2_{\mathcal{G}}(\mathcal{A})$ and $\tilde{\omega}^{(r,s)} \in \Omega^{r,s}_{\mathcal{G}}(\mathcal{A})$ with $r + s = 2$.

As already noted in [5], the only quantity which is both $\mathbf{s}$ and $\bar{\mathbf{s}}$ invariant is $\tilde{\omega}^{1,1}$. The quantity $\tilde{\omega}^{0,2}$ is invariant only under $\bar{\mathbf{s}}$-transformation while $\tilde{\omega}^{2,0}$ is invariant only under s-transformation, i.e.

$$\tilde{\omega}^{1,1} \in H^{1,1}_{\mathcal{G}}(\mathcal{A}), \qquad \tilde{\omega}^{0,2} \in H^{0,2}_{\mathcal{G}}(\mathcal{A}), \qquad \tilde{\omega}^{2,0} \notin H^{2,0}_{\mathcal{G}}(\mathcal{A}). \tag{2.16}$$

The part $(1/4\pi^2)\int_X \mathrm{Tr}(\psi \wedge \bar{\psi}) \wedge \omega^{1,1}$ of $\tilde{\omega}^{1,1}$ is a closed form on $\mathcal{A}$. Then, $\tilde{\omega}^{1,1}$ is the equivariantly closed extension. On the other hand, such an equivariant extension of $\tilde{\omega}^{2,0}$ is not possible since φ, which is the generator of Fun(Lie($\mathcal{G}$)), is of degree $(1, 1)$. This is an example that the Hodge decomposition theorem is not satisfied in the equivariant cohomology. [5]

The TYM theory realizes the Donaldson invariants by expectation values of topological observables [3]. In the $N = 2$ theory, the quantities $\tilde{\omega}^{2,0}$ and $\tilde{\omega}^{0,2}$ are not in the set of observables.

However, it is important to note that $\tilde{\omega}^{2,0}$ and $\tilde{\omega}^{0,2}$ are non-trivial $\mathbf{s}$ and $\bar{\mathbf{s}}$ invariants if they are restricted to the moduli space $\mathcal{M}$ of ASD connections. The Kähler structure on $\mathcal{M}$ guarantees that an s-invariant quantity is $\bar{\mathbf{s}}$-invariant and vice versa. *To put it differently, not all the elements of $H^{*,*}(\mathcal{M})$ can be obtained from the elements of $H^{*,*}_{\mathcal{G}}(\mathcal{A})$ by the restriction and the reduction.* On the other hand, the Donaldson invariants are cup products of (ordinary) cohomology classes on $\mathcal{M}$ evaluated on the fundamental homology cycle of $\mathcal{M}$ provided with a suitable compactification; and the path integral of TYM theory is localized to $\mathcal{M}$. Thus, we should include $\tilde{\omega}^{2,0}$ and $\tilde{\omega}^{0,2}$ to realize the full invariants. Once the localization to $\mathcal{M}$ and a suitable procedure of including $\tilde{\omega}^{2,0}$ and $\tilde{\omega}^{0,2}$ are understood, it is sufficient to consider the $\bar{\mathbf{s}}$-symmetry (that is, the equivariant Dolbeault cohomology $H^{*,*}_{\mathcal{G}}(\mathcal{A})$ only), due to the familiar Hodge decomposition theorem. These are the geometrical reasons underlying the key procedure of Witten's breaking the $N = 2$ supersymmetry down to $N = 1$ symmetry by introducing suitable mass terms [4].

2.5. *The non-Abelian localization*

The $N = 2$ HYM theory is another model for the Donaldson invariants on a Kähler surface [6], adopting Witten's non-Abelian equivariant localization theorem [7]. It shares the same $N = 2$ supersymmetry (or the same structure of the Dolbeault equivariant cohomology) with the $N = 2$ TYM theory. In Ref. [7], Witten proved that the two-dimensional Yang–Mills theory is equivalent to the two-dimensional version of the TYM theory (or the two-dimensional version of the Donaldson theory). The $N = 2$ HYM theory is a natural generalization of Witten's construction to higher-dimensional Kähler manifolds. In the two-dimensional case, the action functional of the Yang–Mills theory is proportional to the

[5] In the non-equivariant cohomology on a Kähler manifold, a d-closed form is automatically ∂ and $\bar{\partial}$ closed as vice versa.

normed square of the moment map $m\mathcal{A} \to \mathrm{Lie}(\mathcal{G})^*$ where $\mathrm{Lie}(\mathcal{G})^*$ is the dual of $\mathrm{Lie}(\mathcal{G})$ [11,7]. In the higher-dimensional Kähler manifolds, one can define a similar moment map $\mathcal{A}^{1,1} \to \mathrm{Lie}(\mathcal{G})^*$ after restricting $\mathcal{A}$ to the space $\mathcal{A}^{1,1}$ of all connections whose curvature two-form is of type $(1,1)$. The classical version of the $N=2$ HYM theory is the Yang–Mills theory restricted to $\mathcal{A}^{1,1}$ whose action functional is also proportional to the norm squared of the moment map [2,6]. A version of the non-Abelian localization theorem states (in a field theoretical context) that a path integral with an action functional given by the normed square of the equivariant moment map of field configuration can be expressed as sums of contributions of the critical points. Such a path integral can be used to obtain cohomology rings of the reduced phase space. [6] The relevant path integrals are the partition function and the expectation value of observables which correspond to equivariantly closed form on the field configuration.

There would be two models of the equivariant localization, the original de Rham model of Witten and the Dolbeault model. The $N=2$ HYM theory is an example of the latter. In terms of the de Rham model of the equivariant localization, the entire (de Rham) cohomology rings on the reduced phase space can in principle be obtained. On the other hand, a new problem arises in the Dolbeault model since not all the Dolbeault cohomology classes on the reduced phase space would be obtained from the elements of the Dolbeault equivariant cohomology.

One of the main purposes of this paper is to eliminate that problem in the $N=2$ HYM theory on the Kähler surface. [7] Clearly, this is closely related to the similar problem of the $N=2$ TYM theory.

3. New construction

In this section, we construct a new $N=2$ TYM theory to overcome the problem of the on-shell invariants. In the new construction, we will impose different transformation laws for anti-ghost multiplets.

3.1. Action

We introduce a commuting anti-ghost $B^0 \in \Omega^0(\mathfrak{g}_E)$ in the adjoint representation with $(U,R)=(-2,0)$. Then (2.8) leads us to multiplet $(B^0, \mathrm{i}\chi^0, -\mathrm{i}\bar{\chi}^0, H^0)$ with transformation laws:

$$\begin{aligned} &\mathrm{s}B^0 = -\mathrm{i}\chi^0, && \mathrm{s}\chi^0 = 0,\\ &\bar{\mathrm{s}}B^0 = \mathrm{i}\bar{\chi}^0, && \bar{\mathrm{s}}\bar{\chi}^0 = 0,\\ &\mathrm{s}\bar{\chi}^0 = H^0 - \tfrac{1}{2}[\varphi, B^0], && \mathrm{s}H^0 = -\tfrac{1}{2}\mathrm{i}[\varphi, \chi^0],\\ &\bar{\mathrm{s}}\chi^0 = H^0 + \tfrac{1}{2}[\varphi, B^0], && \bar{\mathrm{s}}H^0 = -\tfrac{1}{2}\mathrm{i}[\varphi, \bar{\chi}^0]. \end{aligned} \tag{3.1}$$

[6] This is a rapidly growing subject and we will not go into details. Recent developments can be found in [15–19].

[7] On the other hand, the $N=2$ HYM theory on a Riemann surface has no such problem, thus not really different from the original theory [7], since every $\bar{\mathrm{s}}$-closed observables are s-closed [5].

We also introduce an anti-commuting anti-ghost $\chi^{2,0} \in \Omega^{2,0}(\mathfrak{g}_E)$ with $(U, R) = (-1, 1)$ and an anti-commuting anti-ghost $\bar{\chi}^{0,2} \in \Omega^{0,2}(\mathfrak{g}_E)$ with $(U, R) = (-1, -1)$ with transformation laws:

$$\begin{array}{ll} \mathbf{s}\chi^{2,0} = 0, & \mathbf{s}H^{2,0} = -\mathrm{i}[\varphi, \chi^{2,0}], \\ \bar{\mathbf{s}}\chi^{2,0} = H^{2,0}, & \bar{\mathbf{s}}H^{2,0} = 0, \\ \mathbf{s}\bar{\chi}^{0,2} = H^{0,2}, & \mathbf{s}H^{0,2} = 0, \\ \bar{\mathbf{s}}\bar{\chi}^{0,2} = 0, & \bar{\mathbf{s}}H^{0,2} = -\mathrm{i}[\varphi, \bar{\chi}^{0,2}]. \end{array} \tag{3.2}$$

One can easily check that these satisfy the commutation relations (2.8).

Now, the most general form of new $N = 2$ supersymmetric action is

$$S = \mathrm{i}\mathbf{s}\overline{V} + \mathrm{i}\bar{\mathbf{s}}\, V + \tfrac{1}{2}(\mathbf{s}\bar{\mathbf{s}} - \bar{\mathbf{s}}\mathbf{s})\mathcal{B}, \tag{3.3}$$

where V and $\overline{V}$ should be $\bar{\mathbf{s}}$ and $\mathbf{s}$ closed quantities with (U, R) numbers $(-1, 1)$ and $(-1, -1)$, respectively. One finds the following unique choices:

$$\begin{aligned} \overline{V} &= -\frac{1}{h^2}\int_X \mathrm{Tr}\, \bar{\chi}^{0,2} \wedge *F^{2,0}, \\ V &= -\frac{1}{h^2}\int_X \mathrm{Tr}\, \chi^{2,0} \wedge *F^{0,2}, \\ \mathcal{B} &= -\frac{1}{h^2}\int_X \mathrm{Tr}\Big(B^0 f + \alpha\chi^0\bar{\chi}^0\Big)\,\omega^2 - \frac{2\beta}{h^2}\int_X \mathrm{Tr}\,\chi^{2,0} \wedge *\bar{\chi}^{0,2}, \end{aligned} \tag{3.4}$$

where $\alpha, \beta = 0, or\ 1$ and $f = \frac{1}{2}\Lambda F^{1,1}$ where Λ is the operator of contraction with the Kähler form ω. For $\alpha = \beta = 1$, we find

$$\begin{aligned} S = \frac{1}{h^2}\int_X \mathrm{Tr}\Big[& -2H^{2,0} \wedge *H^{0,2} - \mathrm{i}H^{2,0} \wedge *F^{0,2} - \mathrm{i}H^{0,2} \wedge *F^{2,0} \\ & +2\mathrm{i}[\varphi, \chi^{2,0}] \wedge *\bar{\chi}^{0,2} + \mathrm{i}\chi^{2,0} \wedge *\bar{\partial}_A\bar{\psi} + \mathrm{i}\bar{\chi}^{0,2} \wedge *\partial_A\psi \\ & -\Big(2H^0H^0 + 2\mathrm{i}H^0 f - 2\mathrm{i}[\varphi, \chi^0]\bar{\chi}^0 + \tfrac{1}{2}B^0\Lambda\left((\mathrm{i}\partial_A\bar{\partial}_A - \mathrm{i}\bar{\partial}_A\partial_A)\varphi - 2[\psi, \bar{\psi}]\right) \\ & -\tfrac{1}{2}[\varphi, B^0][\varphi, B^0] - \mathrm{i}\bar{\chi}^0\Lambda\bar{\partial}_A\psi - \mathrm{i}\chi^0\Lambda\partial_A\bar{\psi}\Big)\frac{\omega^2}{2!}\Big]. \end{aligned} \tag{3.5}$$

We can integrate out $H^{2,0}$, $H^{0,2}$ and H^0 from the action by setting $H^{2,0} = -\frac{1}{2}\mathrm{i}F^{2,0}$, $H^{0,2} = -\frac{1}{2}\mathrm{i}F^{0,2}$ and $H^0 = -\frac{1}{2}\mathrm{i}f^0$, or by the Gaussian integral, which leads to modified transformation laws

$$\begin{array}{ll} \mathbf{s}\bar{\chi}^{0,2} = -\frac{1}{2}\mathrm{i}F^{0,2}, & \bar{\mathbf{s}}\chi^{2,0} = -\frac{1}{2}\mathrm{i}F^{2,0}, \\ \mathbf{s}\bar{\chi}^0 = -\frac{1}{2}\mathrm{i}f - \frac{1}{2}[\varphi, B^0], & \bar{\mathbf{s}}\chi^0 = -\frac{1}{2}\mathrm{i}f + \frac{1}{2}[\varphi, B^0]. \end{array} \tag{3.6}$$

One can see that the locus of $\mathbf{s}$ and $\bar{\mathbf{s}}$ fixed points in the above transformations is precisely the space of ASD connections. Now we can rewrite the action as

$$
\begin{aligned}
S = \frac{1}{h^2}\int_X \mathrm{Tr}\Big[& -\tfrac{1}{2}F^{2,0}\wedge *F^{0,2} + \mathrm{i}\chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} + \mathrm{i}\bar{\chi}^{0,2}\wedge *\partial_A\psi \\
& -2\mathrm{i}[\varphi,\chi^{2,0}]\wedge *\bar{\chi}^{0,2} \quad - \Big(\tfrac{1}{2}f^2 - 2\mathrm{i}[\varphi,\chi^0]\bar{\chi}^0 - \mathrm{i}\bar{\chi}^0\Lambda\bar{\partial}_A\psi - \mathrm{i}\chi^0\Lambda\partial_A\bar{\psi} \\
& -\tfrac{1}{2}[\varphi,B^0][\varphi,B^0] + \tfrac{1}{2}B^0\Lambda\left((\mathrm{i}\partial_A\bar{\partial}_A - \mathrm{i}\bar{\partial}_A\partial_A)\varphi - 2[\psi,\bar{\psi}]\right)\Big)\frac{\omega^2}{2!}\Big].
\end{aligned} \tag{3.7}
$$

One can easily check that this new theory shares almost all the properties with the old theory studied in [5]. A notable difference between the two theories is that $\chi^{2,0}$ ($\bar{\chi}^{0,2}$) is no longer s-exact ($\bar{\mathrm{s}}$-exact) in the new setting.

Remark. One may wonder why the transformation laws for the anti-ghosts multiplets Eqs. (3.1) and (3.2) are different. To understand this, we should recall the interpretation of TYM theory of Atiyah–Jeffrey [20] based on the Mathai–Quillen formalism [21]. Consider an infinite-dimensional vector bundle $\mathcal{Q}$ over $\mathcal{A}/\mathcal{G}$ whose section s is $s(A) = -F^+(A)$ where F^+ is the self-dual part of the curvature. The moduli space $\mathcal{M}$ of ASD connections is the zero-locus of the section s. In our case, we can decompose the section s (the bundle $\mathcal{Q}$) according to the decompositions $F^+(A) = F^{2,0}(A') \oplus f(A', A'')\omega \oplus F^{0,2}(A'')$. Roughly speaking, the anti-ghosts live in the dual space of the fiber V of $\mathcal{Q}$ [22]. We have introduced the commuting anti-ghost B^0 for the constraint $f(A', A'') = \frac{1}{2}\Lambda F^{1,1}(A', A'') = 0$ and the anti-commuting anti-ghosts $\bar{\chi}^{0,2}$ and $\chi^{2,0}$ for the constraints $F^{2,0}(A') = 0$ and $F^{0,2}(A'') = 0$, respectively. The underlying reason for the different transformation laws, Eqs. (3.1) and (3.2), is that $F^{2,0}(A')$ and $F^{0,2}(A'')$ depend only on A' and A'', respectively, while $f(A', A'')$ depends both on A' and A''. The details are given in [23].

3.2. Fermionic zero-modes

Important properties common to both old and new $N = 2$ topological Yang–Mills theories are the roles of fermionic zero-modes. We will briefly recall the results of [5]. The related mathematical topics can be found in [2,24].

It is convenient to use the language of holomorphic vector bundles. It is well known that an ASD connection A endows E with a holomorphic structure $\mathcal{E}_A$ of given topological type. Let $\mathrm{End}_0(\mathcal{E}_A)$ be the trace-free endomorphism bundle of $\mathcal{E}_A$. It turns out that zero-modes of $\bar{\chi}_0$, $\bar{\psi}$ and $\bar{\chi}^{0,2}$ define elements of $H^0(\mathrm{End}_0(\mathcal{E}_A))$, $H^1(\mathrm{End}_0(\mathcal{E}_A))$ and $H^2(\mathrm{End}_0(\mathcal{E}_A))$, respectively. The formal complex dimension of the moduli space $\mathcal{M}$ is $(-\mathbf{h}^{0,0}+\mathbf{h}^{0,1}-\mathbf{h}^{0,2})$, where $\mathbf{h}^{0,p} = \dim_{\mathbb{C}} H^p(\mathrm{End}_0(\mathcal{E}_A))$.

Since the fermionic zero-modes of $(\bar{\chi}_0, \bar{\psi}, \bar{\chi}^{0,2})$ carry the U-charge $(-1, 1, -1)$, the half of the net violation $\Delta U/2$ of the U-number in the path integral measure is equal to the formal complex dimension. It is important to note that there is no net R-number violation in the path integral measure [5]. We assume, throughout this paper, that there exist the zero-modes pairs of ψ and $\bar{\psi}$ only. Then the moduli space is a smooth Kähler manifold with complex dimension $d = 4k - 3(1 + p_g)$, identical to the number of $\bar{\psi}$ zero-modes.

It is convenient to introduce quantum operators $\hat{U}$ and $\hat{R}$ such that:

$$
\begin{array}{llll}
\hat{U}\chi^0 = u^{-1}\chi^0, & \hat{U}\psi = u\psi, & \hat{U}\chi^{2,0} = u^{-1}\chi^{2,0}, & \hat{U}B^0 = u^{-2}R^0, \\
\hat{U}\bar{\chi}^0 = u^{-1}\bar{\chi}^0, & \hat{U}\bar{\psi} = u\bar{\psi}, & \hat{U}\bar{\chi}^{0,2} = u^{-1}\bar{\chi}^{0,2}, & \hat{R}B^0 = B^0, \\
\hat{R}\chi^0 = r\chi^0, & \hat{R}\psi = r\psi, & \hat{R}\chi^{2,0} = r\chi^{2,0}, & \hat{U}\varphi = u^2\varphi^0, \\
\hat{R}\bar{\chi}^0 = r^{-1}\bar{\chi}^0, & \hat{R}\bar{\psi} = r^{-1}\bar{\psi}, & \hat{R}\bar{\chi}^{0,2} = r^{-1}\bar{\chi}^{0,2}, & \hat{R}\varphi = \varphi.
\end{array}
\tag{3.8}
$$

Then the action S is invariant under the transformations generated by $\hat{U}$ and $\hat{R}$. Now the fermionic part $\mathcal{D}X_f$ of path integral measure, after integrating out every non-zero modes, reduces to

$$
\mathcal{D}\hat{X}_f = \prod_i^d \psi_i\bar{\psi}_i, \tag{3.9}
$$

which transforms, under $\hat{U}$ and $\hat{R}$, as

$$
\mathcal{D}\hat{X}_f \to \mathcal{D}\hat{X}_f u^{-2d}. \tag{3.10}
$$

Thus, the expectation value of topological observables

$$
\left\langle \prod_i^n \tilde{\omega}^{r_i,s_i} \right\rangle = \frac{1}{\mathrm{vol}(\mathcal{G})} \int \mathcal{D}X \mathrm{e}^{-S} \cdot \prod_{i=1}^r \tilde{\omega}^{r_i,s_i}, \tag{3.11}
$$

evaluated with the action S vanishes unless (see [4] for related analysis)

$$
\sum_{i=1}^n (r_i + s_i) = 2d \text{ and } \sum_{i=1}^n (r_i - s_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^r (r_i, s_i) = (d, d). \tag{3.12}
$$

This selection rule is, more or less, identical to the statement that the Donaldson invariants are pure Hodge type of (d, d) [25,26].

3.3. *Including the on-shell observables*

There is a nice method to deal with on-shell invariant quantities [27, pp. 149–151]. To use $\tilde{\omega}^{0,2}$ and $\tilde{\omega}^{2,0}$, we should change the transformation laws (3.2) as

$$
\bar{\mathrm{s}}\bar{\chi}^{0,2} = \frac{h^2}{4\pi^2}\varphi\omega^{0,2}, \qquad \mathrm{s}\chi^{2,0} = \frac{h^2}{4\pi^2}\varphi\omega^{2,0}, \tag{3.13}
$$

and add the terms

$$
-\frac{1}{8\pi^2}\int_X \mathrm{Tr}(\psi \wedge \psi) \wedge \omega^{0,2} - \frac{1}{8\pi^2}\int_X \mathrm{Tr}(\bar{\psi} \wedge \bar{\psi}) \wedge \omega^{2,0} \tag{3.14}
$$

to the action (3.7). Then the action is both s and $\bar{\mathrm{s}}$ invariant with the modified transformation laws of (3.13).

However, we cannot use the above prescription in the old construction. Since φ is both s and $\bar{\mathrm{s}}$ closed, we have $\bar{\mathrm{s}}^2\bar{\chi}^{0,2} = \mathrm{s}^2\chi^{2,0} = 0$. However, Eqs. (2.11) and (3.13) show that $\bar{\mathrm{s}}^2 B^{2,0} = \mathrm{i}\bar{\mathrm{s}}\bar{\chi}^{2,0} \neq 0$. Thus, the changes of the transformation laws as (3.13) do violate the relations $\mathrm{s}^2 = \bar{\mathrm{s}}^2 = 0$. This is why the old theory realizes only the algebraic part of

Donaldson's polynomials, defined by algebraic cycles which are Poincaré dual to elements of $H^{1,1}(S,\mathbb{Z})$.

At this point, it is sufficient to consider only $N = 1$ part of the supersymmetry as explained in Section 2.2. We choose $\bar{\mathrm{s}}$-symmetry. Since $\tilde{\omega}^{0,2}$ is $\bar{\mathrm{s}}$-invariant and $\tilde{\omega}^{2,0}$ is $\bar{\mathrm{s}}$-invariant modulo $\bar{\chi}^{0,2}$-equation of motion, it is sufficient to change the transformation law for $\bar{\chi}^{0,2}$ only in Eq. (3.2) as

$$\bar{\mathrm{s}}\bar{\chi}^{0,2} = \frac{h^2}{4\pi^2}\varphi\omega^{0,2}, \tag{3.15}$$

and add

$$-\frac{1}{8\pi^2}\int_X \mathrm{Tr}(\psi\wedge\psi)\wedge\omega^{0,2} \equiv -\tilde{\omega}^{2,0}, \tag{3.16}$$

to the action (3.7):

$$\begin{aligned} S' = \frac{1}{h^2}\int_X \mathrm{Tr}\Big[& -\tfrac{1}{2}F^{2,0}\wedge *F^{0,2} + \mathrm{i}\chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} + \mathrm{i}\bar{\chi}^{0,2}\wedge *\partial_A\psi \\ & -2\mathrm{i}[\varphi,\chi^{2,0}]\wedge *\bar{\chi}^{0,2} - \Big(\tfrac{1}{2}f^2 - 2\mathrm{i}[\varphi,\chi^0]\bar{\chi}^0 - \mathrm{i}\bar{\chi}^0\Lambda\bar{\partial}_A\psi - \mathrm{i}\chi^0\Lambda\partial_A\bar{\psi} \\ & -\tfrac{1}{2}[\varphi,B^0][\varphi,B^0] + \tfrac{1}{2}B^0\Lambda\left((\mathrm{i}\partial_A\bar{\partial}_A - \mathrm{i}\bar{\partial}_A\partial_A)\varphi - 2[\psi,\bar{\psi}]\right)\Big)\frac{\omega^2}{2!}\Big] \\ & -\frac{1}{8\pi^2}\int_X \mathrm{Tr}(\psi\wedge\psi)\wedge\omega^{0,2}. \end{aligned} \tag{3.17}$$

Note that this action has actually the full $N = 2$ symmetry. Clearly, the action $S' = S - \tilde{\omega}^{2,0}$ is not invariant under the transformations generated by $\hat{U}$ and $\hat{R}$. However, the path integral measure, after integrating out every non-zero modes, is identical to the one defined by the action S, since the additional term does not change the equations of zero-modes. Therefore the partition function $\langle 1\rangle'$ for the action S' can be interpreted as the following expectation value:

$$\langle 1\rangle' = \left\langle \sum_{n=0}^{\infty}\frac{1}{n!}(\tilde{\omega}^{2,0})^n \right\rangle \tag{3.18}$$

evaluated in the theory with the action S. Clearly, this is non-zero only for $d = 0$ and identical to $\langle 1\rangle$.

One can further add the following term to S' maintaining the $\bar{\mathrm{s}}$-symmetry,

$$\begin{aligned} & \bar{\mathrm{s}}\Big(\int_X \mathrm{Tr}(B^0\bar{\chi}^{0,2})\wedge\omega^{2,0}\Big) \\ & = \mathrm{i}\int_X \mathrm{Tr}(\bar{\chi}^0\bar{\chi}^{0,2})\wedge\omega^{2,0} + \frac{h^2}{4\pi^2}\int_X \mathrm{Tr}(B^0\varphi)\omega^{2,0}\wedge\omega^{0,2}. \end{aligned} \tag{3.19}$$

Adding these terms will explicitly break the $N = 2$ supersymmetry down to the $N = 1$ supersymmetry (the $\bar{\mathrm{s}}$-symmetry). The new $\bar{\mathrm{s}}$-invariant action is

$$S'' = S - \frac{1}{8\pi^2}\int\limits_X \mathrm{Tr}(\psi \wedge \psi) \wedge \omega^{0,2} + \mathrm{i}\int\limits_X \mathrm{Tr}(\bar{\chi}^0 \bar{\chi}^{0,2}) \wedge \omega^{2,0}$$
$$+\frac{h^2}{4\pi^2}\int\limits_X \mathrm{Tr}(B^0\varphi)\omega^{2,0} \wedge \omega^{0,2}. \tag{3.20}$$

The above procedures to obtain S' and S'' from the original $N = 2$ supersymmetric action was directly motivated from Section 3 of Ref. [4]. Adding $-\tilde{\omega}^{2,0}$ to the action S gives the bare mass to ψ. Adding (3.19) to S' by breaking the $N = 2$ symmetry down to $N = 1$ induces the mass gap to φ. It is natural due to the supersymmetry $\bar{s}\psi = -\mathrm{i}\partial_A\varphi$. The mass gap can be most easily seen by the B^0-equation of motion for the action S'',

$$(d_A^* d_A + \frac{h^4}{\pi^2} m\bar{m})\varphi + 2\Lambda([\psi, \bar{\psi}]) = 0 \to \langle\varphi\rangle = \frac{-2\Lambda([\psi, \bar{\psi}])}{d_A^* d_A + (h^4/\pi^2)m\bar{m}}, \tag{3.21}$$

where we have used the Kähler identities

$$\bar{\partial}_A^* = \mathrm{i}[\partial_A, \Lambda], \qquad \partial_A^* = -\mathrm{i}[\bar{\partial}_A, \Lambda], \tag{3.22}$$

and set $\omega^{2,0} \wedge \omega^{0,2} = m\bar{m}(\omega \wedge \omega)$. The mass gap of the theory was crucial in Witten's calculation in [4]. Of course, the mass gap disappears in the vanishing locus of $\omega^{0,2}$.

4. Deformations to holomorphic Yang–Mills theories

We now turn to HYM theory. Since the terms which are proportional to the Kähler form are identical in the old and new actions S_{old} and S, we can repeat the procedure in [6] to obtain $N = 2$ HYM theory. It is convenient to choose delta function gauge by setting $\alpha = \beta = 0$ in (3.4). Now the action for $N = 2$ TYM theory is

$$S = \frac{1}{h^2}\int\limits_X \mathrm{Tr}\Big[-\mathrm{i}H^{2,0} \wedge *F^{0,2} - \mathrm{i}H^{0,2} \wedge *F^{2,0} + \mathrm{i}\chi^{2,0} \wedge *\bar{\partial}_A\bar{\psi}$$
$$+\mathrm{i}\bar{\chi}^{0,2} \wedge *\partial_A\psi - \Big(2\mathrm{i}H^0 f - \mathrm{i}\bar{\chi}^0\Lambda\bar{\partial}_A\psi - \mathrm{i}\chi^0\Lambda\partial_A\bar{\psi}$$
$$+\tfrac{1}{2}B^0\Lambda\left((\mathrm{i}\partial_A\bar{\partial}_A - \mathrm{i}\bar{\partial}_A\partial_A)\varphi - 2[\psi, \bar{\psi}]\right)\Big)\frac{\omega^2}{2!}\Big]. \tag{4.1}$$

Then, the action of $N = 2$ HYM theory becomes

$$S_H = \frac{1}{h^2}\int\limits_X \mathrm{Tr}\Big[-\mathrm{i}H^{2,0} \wedge *F^{0,2} - \mathrm{i}H^{0,2} \wedge *F^{2,0}$$
$$+\mathrm{i}\chi^{2,0} \wedge *\bar{\partial}_A\bar{\psi} + \mathrm{i}\bar{\chi}^{0,2} \wedge *\partial_A\psi\Big]$$
$$-\frac{1}{4\pi^2}\int\limits_X \mathrm{Tr}(\mathrm{i}\varphi F + \psi \wedge \bar{\psi}) \wedge \omega - \frac{\varepsilon}{8\pi^2}\int\limits_X \mathrm{Tr}(\varphi^2)\frac{\omega^2}{2!}. \tag{4.2}$$

This is equivalent to the action studied in [6]. The difference is that $\chi^{2,0}$ and $\bar{\chi}^{0,2}$ are no longer BRST exact in this new setting.

Since $N=2$ HYM theory has the same $N=2$ supersymmetry and the same topological observables as those of $N=2$ TYM theory, we can repeat the same procedure to deal with the on-shell invariant quantities. It is sufficient to consider the $N=1$ part of the symmetry and we, once again, consider the $\bar{\mathrm{s}}$-symmetry. Adding (3.16) to the action S_H, we have a new action

$$\begin{aligned} S'_H = &\frac{1}{h^2}\int_X \mathrm{Tr}\Big[\mathrm{i}H^{2,0}\wedge *F^{0,2} - \mathrm{i}H^{0,2}\wedge *F^{2,0} + \mathrm{i}\chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} + \mathrm{i}\bar{\chi}^{0,2}\wedge *\partial_A\psi\Big] \\ &-\frac{1}{4\pi^2}\int_X \mathrm{Tr}(\mathrm{i}\varphi F + \psi\wedge\bar{\psi})\wedge\omega - \frac{\varepsilon}{8\pi^2}\int_X \mathrm{Tr}(\varphi^2)\frac{\omega^2}{2!} \\ &-\frac{1}{8\pi^2}\int_X \mathrm{Tr}(\psi\wedge\psi)\wedge\omega^{0,2}, \end{aligned} \tag{4.3}$$

where the change of the transformation of $\bar{\mathrm{s}}$ as (3.15) is understood. Of course, we start from the action S' (in the delta function gauge) and then define the mapping to the HYM theory. Both procedures give the identical result.

The partition function $Z(\varepsilon)_d$ of the HYM theory with action S_H is a generating functional

$$Z(\varepsilon)_d = \sum_{r,s}^{r+2s=d} \frac{\varepsilon^s}{r!s!}\left\langle \tilde{\omega}^r \Theta^s\right\rangle + \mathrm{O}(\mathrm{e}^{-c/\varepsilon}), \tag{4.4}$$

where

$$\begin{aligned} \tilde{\omega} &= \frac{1}{4\pi^2}\int_X \mathrm{Tr}(\mathrm{i}\varphi F + \psi\wedge\bar{\psi})\wedge\omega, \\ \Theta &= \frac{1}{8\pi^2}\int_X \mathrm{Tr}(\varphi^2)\frac{\omega^2}{2!}, \end{aligned} \tag{4.5}$$

and c in exponentially small terms is the positive minimum value of $-(1/2\pi^2)\int_X \frac{1}{2}\omega^2\mathrm{Tr}f_c^2$ for the higher critical points f_c [7,6]. Note that the partition function $Z'(\varepsilon)_d$ with action S'_H is identical to $Z(\varepsilon)_d$:

$$\begin{aligned} Z'(\varepsilon)_d = \left\langle \sum_{n=0}^{\infty}\frac{1}{n!}(\tilde{\omega}^{2,0})^n\right\rangle_H &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r,s}^{r+2s=d-n}\frac{\varepsilon^s}{r!s!}\left\langle (\tilde{\omega}^{2,0})^n\tilde{\omega}^r\Theta^s\right\rangle + \cdots \\ &= \sum_{r,s}^{r+2s=d}\frac{\varepsilon^s}{r!s!}\left\langle\tilde{\omega}^r\Theta^s\right\rangle + \cdots \\ &= Z(\varepsilon)_d. \end{aligned} \tag{4.6}$$

However, the HYM theories with the actions S_H and S'_H have different localizations. The $H^{2,0}$, $H^{0,2}$, $\chi^{2,0}$ and $\bar{\chi}^{0,2}$ integrations localize both the theories to $T\mathcal{A}^{1,1}$,

$$F^{2,0} = F^{0,2} = \partial_A \psi = \bar{\partial}_A \bar{\psi} = 0. \tag{4.7}$$

The φ equations of the motion for both the actions S_H and S'_H give

$$2\mathrm{i} f + \varepsilon\varphi = 0. \tag{4.8}$$

Then, a fixed point equation of the basic supersymmetry leads to

$$\bar{\mathrm{s}}\psi = -\mathrm{i}\partial_A\varphi = 0 \Longrightarrow \partial_A f = 0 \Longleftrightarrow d_A f = 0. \tag{4.9}$$

However, the theory with the action S'_H has an additional fixed point equation due to Eq. (3.15)

$$\bar{\mathrm{s}}\bar{\chi}^{0,2} = (h^2/4\pi^2)\varphi\omega^{0,2} = 0. \tag{4.10}$$

This shows that in the vanishing locus $C \subset X$ of $\omega^{0,2}$ the same localization governs the two theories, while in the complement of C the theory with the action S'_H is localized to the instanton ($f = 0$). Due to the supersymmetry, we also have

$$f = 0 \Longrightarrow \bar{\mathrm{s}} f = \bar{\partial}^*_A \bar{\psi} = 0. \tag{4.11}$$

Finally, we note that the HYM theory with action S'_H is entirely equivalent to the TYM theory with action S' for hyper-Kähler surfaces. There will be no contributions of higher critical points. Since those manifolds have only one holomorphic harmonic two-form which is nowhere vanishing, the fixed point equation (4.10) leads to $\varphi = 0$. Then, Eq (4.8) implies that there will be no contributions of higher critical points. This may be related to a general fact that the twisting of $N = 2$ supersymmetric theory does not change anything on a manifold with trivial canonical line bundle [4,27].

4.1. *Deformation from the action S''*

One can also start with the TYM theory with action S'' (in the delta function gauge) which has the $\bar{\mathrm{s}}$-symmetry only. We will show that there is a suitable deformation of S'' to the HYM theory with action S'_H.

We can add to the action S'' an $\bar{\mathrm{s}}$-exact term maintaining the $\bar{\mathrm{s}}$-symmetry,

$$\mathrm{i}\bar{\mathrm{s}}\left(-\frac{4t}{h^2}\int_X \frac{\omega^2}{2!}\mathrm{Tr}(B^0\chi^0)\right) = -\frac{4t}{h^2}\int_X \frac{\omega^2}{2!}\mathrm{Tr}(\chi^0\bar{\chi}^0 + \mathrm{i}H^0B^0), \tag{4.12}$$

which leads to a family of $\bar{\mathrm{s}}$-symmetric action $S''(t)$

$$S''(t) = S'' + \mathrm{i}\bar{\mathrm{s}}\left(-\frac{4t}{h^2}\int_X \frac{\omega^2}{2!}\mathrm{Tr}(B^0\chi^0)\right)$$
$$= \frac{1}{h^2}\int_X \mathrm{Tr}\Big[-\mathrm{i}H^{2,0} \wedge *F^{0,2} - \mathrm{i}H^{0,2} \wedge *F^{2,0} + \mathrm{i}\chi^{2,0} \wedge *\bar{\partial}_A\bar{\psi} + \mathrm{i}\bar{\chi}^{0,2} \wedge *\partial_A\psi$$

$$
-\Big(2\mathrm{i}H^0(f+2tB^0)+\tfrac{1}{2}B^0\Lambda\left((\mathrm{i}\partial_A\bar\partial_A-\mathrm{i}\bar\partial_A\partial_A)\varphi-2[\psi,\bar\psi]\right)+4t\chi^0\bar\chi^0
$$
$$
-\mathrm{i}\bar\chi^0\Lambda\bar\partial_A\psi-\mathrm{i}\chi^0\Lambda\partial_A\bar\psi\Big)\frac{\omega^2}{2!}\Big]-\frac{1}{8\pi^2}\int_X\mathrm{Tr}(\psi\wedge\psi)\wedge\omega^{0,2}
$$
$$
+\mathrm{i}\int_X\mathrm{Tr}(\bar\chi^0\bar\chi^{0,2})\wedge\omega^{2,0}+\frac{h^2}{4\pi^2}\int_X\mathrm{Tr}(B^0\varphi)\omega^{2,0}\wedge\omega^{0,2}. \tag{4.13}
$$

After integrating B^0, χ^0 and $\bar\chi^0$ out, we have

$$
S''(t)=\frac{1}{h^2}\int_X\mathrm{Tr}\Big[-\mathrm{i}H^{2,0}\wedge *F^{0,2}-\mathrm{i}H^{0,2}\wedge *F^{2,0}+\mathrm{i}\chi^{2,0}\wedge *\bar\partial_A\bar\psi
$$
$$
+\mathrm{i}\bar\chi^{0,2}\wedge *\partial_A\psi\;+\left(\frac{1}{4t}fd_A^*d_A\varphi-\frac{1}{2t}f\Lambda([\psi,\bar\psi])+\frac{1}{4t}(\partial_A^*\psi)(\bar\partial_A^*\bar\psi)\right)\frac{\omega^2}{2!}\Big]
$$
$$
-\frac{1}{8\pi^2}\int_X\mathrm{Tr}(\psi\wedge\psi)\wedge\omega^{0,2}-\frac{1}{4t}\int_X\mathrm{Tr}(\bar\partial_A^*\bar\psi\,\bar\chi^{0,2})\wedge\omega^{2,0}
$$
$$
-\frac{h^2}{8\pi^2 t}\int_X\mathrm{Tr}(\varphi f)\omega^{2,0}\wedge\omega^{0,2}, \tag{4.14}
$$

where we have used the Kähler identities (3.22).

Now we examine what kind of localization governs the deformed theory with action $S''(t)$. The $H^{2,0}$ and $H^{0,2}$ integration localize the theory to $\mathcal{A}^{1,1}$. The φ-integration gives

$$
-d_A^*d_Af\frac{\omega^2}{2}-\frac{h^4}{2\pi^2}f\omega^{2,0}\wedge\omega^{0,2}=0
$$
$$
\Longrightarrow-\int_X\mathrm{Tr}(d_Af\wedge *d_Af)-\frac{h^4}{\pi^2}\int_X m\bar m\mathrm{Tr}(f*f)=0. \tag{4.15}
$$

Thus, the fixed points of the deformed theory are $d_Af=0$ at the vanishing locus C of $\omega^{0,2}$, while it is instanton ($f=0$) in the complement of C. We see that the deformed theory has the same bosonic fixed point with the HYM theory with the action S'_H. The $\chi^{2,0}$ and $\bar\chi^{0,2}$ integrations give

$$
\mathrm{i}\bar\partial_A\bar\psi=0,\qquad \mathrm{i}\partial_A\psi+\frac{h^2}{4t}\bar\partial_A^*\bar\psi\wedge\omega^{2,0}=0. \tag{4.16}
$$

In the locus C, the above equations reduce to

$$
\partial_A\psi=0,\qquad \bar\partial_A\bar\psi=0, \tag{4.17}
$$

while in the complement of C we have additional equation

$$
\bar\partial_A^*\bar\psi=0. \tag{4.18}
$$

This coincides to the bosonic fixed point $f = 0$ in the complement of C

$$\bar{\mathrm{s}} f = 0 \Longrightarrow \bar{\partial}_A^* \bar{\psi} = 0. \tag{4.19}$$

Thus, the deformed theory has the same fixed points with the HYM theory with action S'_H. Then, the final step of the deformation is to consider the expectation value of the observable $\exp(\tilde{\omega} + \varepsilon\Theta)$ with $t = \infty$ limit, which leads to the action S'_H.

It is interesting to note that the action S'_H actually has the full $N = 2$ symmetry. During the deformation of the $N = 1$ symmetric TYM action S'' to the HYM theory, the broken $N = 1$ symmetry (the s-symmetry) is restored. We do not know whether this has any physical application.

Anyway, it is sufficient to consider the $\bar{\mathrm{s}}$-symmetry only. If we want to maintain the full symmetry explicitly, we should change the transformation laws as (3.13) and add (3.14) to the action S_H, which leads to

$$S''_H = S_H - \tilde{\omega}^{2,0} - \tilde{\omega}^{0,2}. \tag{4.20}$$

The partition function $Z''(\varepsilon)$ with the action S''_H is identical to

$$Z''(\varepsilon) = \left\langle \sum \frac{1}{n!} (\tilde{\omega}^{0,2})^n \right\rangle'_H = \left\langle \sum \frac{1}{n!n!} (\tilde{\omega}^{0,2} \tilde{\omega}^{0,2})^n \right\rangle_H, \tag{4.21}$$

where $\langle\cdot\rangle'_H$ denotes the expectation value evaluated with the action S'_H. In the above identification, considering $\bar{\mathrm{s}}$-symmetry only is understood such that $\tilde{\omega}^{0,2}$ can be an observable.

4.2. A simple calculation

Now we determine the Donaldson polynomial invariants on $H^{0,2}(X, \mathbb{Z}) \oplus H^{2,0}(X, \mathbb{Z})$. We consider the partition function $Z'(\varepsilon)_d$ of HYM theory with action S'_H:

$$\begin{aligned} Z'(\varepsilon)_d = {} & \frac{1}{\mathrm{vol}(\mathcal{G})} \int \mathcal{D}A' \mathcal{D}A'' \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\varphi \mathcal{D}H^{2,0} \mathcal{D}H^{0,2} \mathcal{D}\chi^{2,0} \mathcal{D}\bar{\chi}^{0,2} \\ & \times \exp\biggl(\frac{1}{h^2} \int_X \mathrm{Tr}\Bigl[\mathrm{i}H^{2,0} \wedge *F^{0,2} + \mathrm{i}H^{0,2} \wedge *F^{2,0} \\ & -\mathrm{i}\chi^{2,0} \wedge *\bar{\partial}_A \bar{\psi} - \mathrm{i}\bar{\chi}^{0,2} \wedge *\partial_A \psi \Bigr] \\ & + \frac{1}{4\pi^2} \int_X \mathrm{Tr}\bigl(\mathrm{i}\varphi F^{1,1} + \psi \wedge \bar{\psi}\bigr) \wedge \omega \\ & + \frac{\varepsilon}{8\pi^2} \int_X \frac{\omega^2}{2!} \mathrm{Tr}\,\varphi^2 + \frac{1}{8\pi^2} \int_X \mathrm{Tr}(\psi \wedge \psi) \wedge \omega^{0,2} \biggr). \end{aligned} \tag{4.22}$$

It is more convenient to represent $\tilde{\omega}^{0,2}$ and $\tilde{\omega}^{2,0}$ by

$$\tilde{\omega}^{0,2} = \frac{1}{8\pi^2} \int_{\Gamma} \mathrm{Tr}(\bar{\psi} \wedge \bar{\psi}), \qquad \tilde{\omega}^{2,0} = \frac{1}{8\pi^2} \int_{\bar{\Gamma}} \mathrm{Tr}(\psi \wedge \psi), \tag{4.23}$$

where Γ and $\bar{\Gamma}$ denote homology cycles Poincaré dual to $\omega^{2,0}$ and $\omega^{0,2}$, respectively. Now we want to determine the expectation value $\left\langle(\tilde{\omega}^{0,2})^m\right\rangle'_H$ evaluated in the HYM theory with action S'_H

$$\begin{aligned}\left\langle(\tilde{\omega}^{0,2})^m\right\rangle'_H &= \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle(\tilde{\omega}^{0,2})^m(\tilde{\omega}^{2,0})^n\right\rangle_H \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r,s}^{r+2s=d-n-m}\frac{\varepsilon^s}{r!s!}\left\langle(\tilde{\omega}^{0,2})^m(\tilde{\omega}^{2,0})^n\tilde{\omega}^r\Theta^s\right\rangle+\cdots \\ &= \frac{1}{m!}\sum_{r,s}^{r+2s=d-2m}\frac{\varepsilon^s}{r!s!}\left\langle(\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m\tilde{\omega}^r\Theta^s\right\rangle+\cdots \\ &= \frac{1}{m!}\left\langle(\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m\right\rangle_H.\end{aligned} \tag{4.24}$$

Thus, we consider

$$\begin{aligned}\left\langle(\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m\right\rangle_H &= \frac{1}{\mathrm{vol}(\mathcal{G})}\int\mathcal{D}A'\,\mathcal{D}A''\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\varphi\cdots \\ &\times\exp\left(\cdots+\frac{1}{4\pi^2}\int_X\mathrm{Tr}\big(\mathrm{i}\varphi F^{1,1}+\psi\wedge\bar{\psi}\big)\wedge\omega+\frac{\varepsilon}{8\pi^2}\int_X\frac{\omega^2}{2!}\mathrm{Tr}\,\varphi^2\right) \\ &\times\left(\frac{1}{8\pi^2}\int_{\bar{\Gamma}}\mathrm{Tr}\,\psi\wedge\psi\cdot\frac{1}{8\pi^2}\int_{\Gamma}\mathrm{Tr}\,\bar{\psi}\wedge\bar{\psi}\right)^m.\end{aligned} \tag{4.25}$$

We note that ψ and $\bar{\psi}$ are coupled as free fields with the trivial propagator,

$$\langle\psi_i^a(x)\bar{\psi}_{\bar{j}}^b(y)\rangle = -\mathrm{i}4\pi^2\varepsilon_{i\bar{j}}\delta^{ab}\delta^4(x-y). \tag{4.26}$$

To be more precise, this amounts to perform Gaussian integrals in the action S''_H. The ψ and $\bar{\psi}$ are obviously coupled as free field for the vanishing locus C of $\omega^{0,2}$ in X. The actual calculation (using the Kähler identities) shows that they are coupled as free field even in the complement of C if $\bar{\partial}_A^*\bar{\psi} = \partial_A^*\psi = 0$ which are guaranteed, as explained in Section 4.1. Upon performing the ψ and $\bar{\psi}$ integral, we see (4.25) is equivalent to

$$\begin{aligned}\left\langle\tilde{\omega}^{0,2}\tilde{\omega}^{2,0}\right\rangle_H = \frac{1}{\mathrm{vol}(\mathcal{G})}&\int\mathcal{D}A'\,\mathcal{D}A''\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\varphi\cdots \\ &\times\exp\left(\cdots+\frac{1}{4\pi^2}\int_X\mathrm{Tr}\big(\mathrm{i}\varphi F^{1,1}+\psi\wedge\bar{\psi}\big)\wedge\omega+\frac{\varepsilon}{8\pi^2}\int_X\frac{\omega^2}{2!}\mathrm{Tr}\,\varphi^2\right) \\ &\times m!(\bar{\Gamma}\cdot\Gamma)^m,\end{aligned} \tag{4.27}$$

where $\Gamma \cdot \bar{\Gamma} = \int_X \omega^{0,2} \wedge \omega^{2,0}$ denotes the intersection number. [8] Thus we have the following factorization:

$$\left\langle (\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m \right\rangle_H = \sum_{r,s}^{r+2s=d-2m} \frac{\varepsilon^s}{r!s!} \left\langle (\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m \tilde{\omega}^r \Theta^s \right\rangle + \cdots$$
$$= m! \sum_{r,s}^{r+2s=d-2m} \frac{\varepsilon^s}{r!s!} \langle \tilde{\omega}^r \Theta^s \rangle (\bar{\Gamma} \cdot \Gamma)^m + \cdots, \tag{4.28}$$

that is,

$$\sum_{r,s}^{r+2s=d-2m} \left\langle (\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m \tilde{\omega}^r \Theta^s \right\rangle = m! \sum_{r,s}^{r+2s=d-2m} \langle \tilde{\omega}^r \Theta^s \rangle (\bar{\Gamma} \cdot \Gamma)^m. \tag{4.29}$$

If $d = 2m$, we have

$$\left\langle (\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m \right\rangle = m!(\bar{\Gamma} \cdot \Gamma)^m \langle 1 \rangle. \tag{4.30}$$

Then

$$\left\langle (\tilde{\omega}^{0,2} + \tilde{\omega}^{2,0})^d \right\rangle = \frac{(2m)!}{m!m!} \left\langle (\tilde{\omega}^{0,2}\tilde{\omega}^{2,0})^m \right\rangle = \frac{(2m)!}{m!} (\bar{\Gamma} \cdot \Gamma)^m \langle 1 \rangle. \tag{4.31}$$

4.3. General remarks on the algebraic part

Let M be a *simple* simply connected four-manifold with $b_2^+(M) \geq 3$. Let $q_d(M)$ denote the $SU(2)$ polynomials on $H_0(M, \mathbb{Z}) \oplus H_2(M, \mathbb{Z})$, where $d = 4k - \frac{3}{2}(1 + b_2^+)$. Kronheimer and Mrowka [28] have announced that the Donaldson series $q(M) = \sum_d q_d(X)/d!$ is given by

$$q(M) = e^{Q/2} \sum_{i=1}^{n} a_i e^{K_i}, \tag{4.32}$$

where Q is the intersection form, regarded as a quadratic function ($Q \in \mathrm{Sim}^2(H^2(M, \mathbb{Z}))$), of M, $K_i \in H_2(M)$ denote the simple classes and a_i are non-zero rational numbers. Since Q is a homeomorphism invariant, any relevant information for smooth structures is contained in K_i and a_i.

Recently, Brusse proved that the basic classes K_i are of the type (1, 1), i.e. $K_i \in H^{1,1}(X, \mathbb{Z})$, for a *simple* simply connected algebraic surfaces X with $p_g(X) \geq 1$ [29], using the pureness of the Donaldson invariants for simply connected algebraic surfaces [25]. Then, one of his corollary that for all $\omega^{0,2} \in H^{0,2}(X, \mathbb{Z})$

$$q(\omega^{0,2} + \omega^{2,0}) = q_0 \exp\left[\int \omega^{0,2} \wedge \omega^{2,0}\right], \tag{4.33}$$

[8] Actually, the above Eq. (4.27) should contain a group theoretical factor due to the trace. Since we are dealing with $SU(2)$ case, the omitted factor is $\dim(SU(2))^m = 3^m$. It seems to us that mathematicians usually omit this term.

where q_0 is Donaldson's polynomial of degree zero, can be immediately followed from (4.32). This result says that the algebraic part of Donaldson's polynomials, i.e. the polynomials defined by Li [8], contains as much information as the full polynomials for a *simple* simply connected algebraic surface.

More recently, Witten has shown that all compact Kähler surfaces with $p_g \geq 1$ are of simple type [4]. His completely explicit formula for the full polynomials also imply that all the simple classes (or we should say the Kronheimer–Mrowka–Witten classes) are of the type $(1,1)$, in fact, they are linear combinations of components of the canonical divisor.

Our heuristic calculation shows that for every simply connected compact Kähler surface X with $p_g(X) \geq 1$ and for all $\omega^{0,2} \in H^{0,2}(X,\mathbb{Z})$

$$q(\omega^{0,2}+\omega^{2,0}) = q_0 \exp\left[\int \omega^{0,2}\wedge\omega^{2,0}\right]. \tag{4.34}$$

That is, Eq. (4.31) can be written as

$$q_d(\omega^{2,0}+\omega^{0,2}) = q_0 \frac{(2m)!}{m!}\left(\int_X \omega^{2,0}\wedge\omega^{0,2}\right), \tag{4.35}$$

where $q_0 = \langle 1\rangle$. Thus we have

$$q(\omega^{2,0}+\omega^{0,2}) = q_0 \exp\left[\int_X \omega^{2,0}\wedge\omega^{0,2}\right]. \tag{4.36}$$

All the relevant information (beyond the classical invariants) of Donaldson's polynomial invariants are contained in the algebraic part. We should be able to evaluate the following topological correlation function evaluated by the action functional S'_H (4.3)

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}A'\,\mathcal{D}A''\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\varphi\,\mathcal{D}H^{2,0}\,\mathcal{D}H^{0,2}\,\mathcal{D}\chi^{2,0}\,\mathcal{D}\bar{\chi}^{0,2}\\
&\quad\times\exp\left(\frac{1}{h^2}\int_X \mathrm{Tr}\left[\mathrm{i}H^{2,0}\wedge * F^{0,2}+\mathrm{i}H^{0,2}\wedge * F^{2,0}\right.\right.\\
&\quad\left.-\mathrm{i}\chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi}-\mathrm{i}\bar{\chi}^{0,2}\wedge *\partial_A\psi\right]\\
&\quad+\frac{1}{4\pi^2}\int_X \mathrm{Tr}(\mathrm{i}\varphi F^{1,1}+\psi\wedge\bar{\psi})\wedge\omega+\frac{\varepsilon}{8\pi^2}\int_X\frac{\omega^2}{2!}\mathrm{Tr}\,\varphi^2\\
&\quad\left.+\frac{1}{8\pi^2}\int_X \mathrm{Tr}(\psi\wedge\psi)\wedge\omega^{0,2}\right)\prod_{i=1}^{2m}\frac{1}{4\pi^2}\int_{\Sigma_i}\mathrm{Tr}(\mathrm{i}\varphi F^{1,1}+\psi\wedge\bar{\psi}),
\end{aligned} \tag{4.37}$$

where Σ_i are algebraic cycles Poincaré dual to elements of $H^{1,1}(X,\mathbb{Z})$. In evaluating this expectation value, the term $\tilde{\omega}^{2,0}$ itself do not contribute to the path integral. However, the modification of the transformation law of $\bar{\chi}^{0,2}$ given by Eq. (3.11) dramatically changes the fixed points of the theory.

The path integral of a cohomological field theory with global fermionic symmetry Q is localized to an Q-invariant neighbourhood of the fixed point locus of Q. One must perform the path integral along the fixed point locus exactly, while the transverse path integral can be done in one-loop approximation [30].

A fixed point of the HYM with action S_H is

$$d_A\varphi = 0, \tag{4.38}$$

where $A \in \mathcal{A}^{1,1}$. Thus φ at the fixed point locus is a covariant constant. There can be two branches: (a) if Eq. (4.38) has no non-trivial solutions, that is $\varphi = 0$, the connection A is irreducible; (b) if $\varphi \neq 0$ solves Eq. (4.38), a holomorphic connection A should be reducible and the bundle $\mathcal{E}_A$ splits as a direct sum of holomorphic line bundles $\mathcal{E}_A = L \oplus L^{-1}$. It is worthwhile to note that all higher critical points of HYM theory are reducible holomorphic connections and there are no reducible instantons for generic choices of metric. The HYM theory with action S'_H has additional source of fixed point

$$\bar{s}\bar{\chi}^{0,2} = (h^2/4\pi^2)\varphi(x)\omega^{0,2}(x) = 0. \tag{4.39}$$

Let $C \subset X$ be the locus of $\omega^{0,2}(x) = 0$ and $C^c \subset X$ be the locus of $\omega^{0,2}(x) \neq 0$. Eq. (4.39) forces that $\varphi(x)$ should vanish if $x \in C^c$. On the other hand, $\varphi(x)$ can be either zero or non-zero covariant constant if $x \in C$. That is, we have actually three different branches: (i) *branch* I: If $x \in C^c$, $\varphi(x) = 0$; (ii) *branch* IIa: $x \in C$ and $\varphi(x) = 0$; (iii) *branch* IIb: $x \in C$ and $\varphi(x) \neq 0$. Thus, the path integral (4.37) can be formally written as product of the contributions $P(\mathrm{I})$ and $P(\mathrm{II})$ of the branches I and II, respectively,

$$P(\mathrm{I})P(\mathrm{II}) = P(\mathrm{I})P(\mathrm{IIa}) + P(\mathrm{I})P(\mathrm{IIb}). \tag{4.40}$$

We can evaluate the first part $P(\mathrm{I})P(\mathrm{IIa})$ of the path integral using a similar method adapted in Section 4.2. Note that the higher critical points do not contribute to this path integral, since $\varphi = 0$ at the fixed points in branches I and IIa. For simplicity, we consider $d = 2m$. We can simply set $\varphi = 0$ in (4.37), which leads to

$$\begin{aligned}&\frac{1}{\mathrm{vol}(\mathcal{G})}\int_{T\mathcal{A}^{1,1}} \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp\biggl(\frac{1}{4\pi^2}\int_X \mathrm{Tr}(\psi\wedge\bar{\psi})\wedge\omega\biggr)\\ &\times\frac{1}{4\pi^2}\int_{\Sigma_1}\mathrm{Tr}(\psi\wedge\bar{\psi})\cdots\frac{1}{4\pi^2}\int_{\Sigma_{2m}}\mathrm{Tr}(\psi\wedge\bar{\psi}).\end{aligned} \tag{4.41}$$

Now the Gaussian integral over ψ and $\bar{\psi}$ using (4.26) immediately gives

$$\frac{1}{\mathrm{vol}(\mathcal{G})}\int_{T\mathcal{A}^{1,1}} \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp\biggl(\frac{1}{4\pi^2}\int_X \mathrm{Tr}(\psi\wedge\bar{\psi})\wedge\omega\biggr)Q^{(m)}(\Sigma_1,\ldots,\Sigma_{2m}), \tag{4.42}$$

where $Q^{(m)}$ is a multi-linear form [2] on $H_2(X)$ defined by

$$\begin{aligned}&Q^{(m)}(\Sigma_1,\ldots,\Sigma_{2m})\\ &= \frac{1}{2^m m!}\sum_{\sigma\in S_{2m}} Q(\Sigma_{\sigma(1)},\Sigma_{\sigma(2)})\times\cdots\times Q(\Sigma_{\sigma(2m-1)},\Sigma_{\sigma(2m)}),\end{aligned} \tag{4.43}$$

and Q is the intersection form of X. In particular, if we consider a simply connected hyper-Kähler surface such that $\omega^{0,2} \in H^{0,2}(X, \mathbb{Z})$ is nowhere vanishing, then, only branch I contributes ($C^c \equiv X$) and we can set $d = 2m$. We have

$$q_{2m}(\Sigma_1, \ldots, \Sigma_{2m}) = Q^{(m)}(\Sigma_1, \ldots, \Sigma_{2m}), \tag{4.44}$$

which coincides to the known mathematical answer [2,31].

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